hep-th/0109155
TIT/HEP-471
September, 2001

Dualities between K3 fibered Calabi–Yau three-folds

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Abstract

We propose a way to examine N=1 and N=2 string dualities on Calabi-Yau three-folds and their extensions. Our way is to find out or to construct two types of toric representations of a Calabi-Yau three-fold, which contain phases topologically equivalent or phases connected by flops. We discuss how to find relations among Calabi-Yau three-folds realized in different toric representations. We examine several examples of Calabi-Yau three-folds that have the Hodge numbers, $(h^{1,1}, h^{2,1}) = (5, 185)$ and the various numbers of K3 fibers. We observe that each phase of our examples contains Del Pezzo 4-cycles, B_8 in six ways.

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1 Introduction

A motivation of our work is to examine N=1 and N=2 string dualities from identification of Calabi-Yau three-folds (CY3s). We propose to utilize several different types of toric representations, i.e., local coordinates of a CY3, which are topologically equivalent.

There are two points that characterize a toric representation in the above case: one is the existence of extra tensor multiplets in 6-dimensional intermediate stage and the other the existence of double K3 fibrations in CY3s, which may not be seen clearly from a single K3 fibered representation without using the method given by [1, 2] [3].

First, we use the heterotic-type IIA string duality, that is, if a CY3 admits both K3 and $\mathbf{T}^2$ fibrations with at least one section then type IIA string on the CY3 is dual to a heterotic string on K3$\times\mathbf{T}^2$ [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

Second, we use the heterotic-heterotic string duality, that is, if there are double K3 fibered CY3s then there are two heterotic string compactifications depending on which K3 fibrations are used in the compactification[16]. Furthermore, we can extend to the heterotic-heterotic string duality between two toric representations with the topologically equivalent CY3s.

Third, we use the heterotic-type IIB string duality to examine how to relate type IIA side to the heterotic string sides in the strong or weak coupling regions by calculating discriminants of a mirror polynomial of CY3 in our case and taking an appropriate parameter limit. They correspond to the gauge symmetry of heterotic string side which comes from the four contributions: (one K3 fibration)$\times$(the other K3 fibration) $\times$ (d=6 tensor) $\times$ (further T^2 compactification).

By changing a parameter to another parameter in the mirror polynomial, we can exchange the weak coupling region with a strong coupling region. For a toric representation of a CY3 with single K3 fibration phase, we can see only one-side contribution of K3 fibrations, i.e., weak coupling region only or strong

[3] $\sharp$ of K3 fibrations in each CY3 phases means the number of divisors, J_i with $c_2 \cdot J = 24$ under our basis. These CY3s have the possibility to have other K3 fibers which can not be seen as a sub dual polyhedron. If one take another base then this number may change.

coupling region only. By identifying a double K3 fibered CY3 phase with a single K3 fibered CY3 phase, we can identify strong coupling region physics which is not seen clearly with the weak coupling region physics

We examine two toric representations with the same Hodge numbers, $(h^{1,1}, h^{1,2}) = (5, 185)$ in two series, (III) of **CFPR** model and (IV) of **HLY** model mainly [4] which may satisfy above three dualities and show their relations. Furthermore, $\Delta n_T \geq 3$ case is the touchstone of the identification of (III) and (IV), since the properties of (III) and (IV) are different, i.e., the ADE type singularity appears in (III) and no ADE type singularity does in (IV). The relations of two toric representations with the same Hodge numbers, $(h^{1,1}, h^{1,2}) = (8, 164)$ in (III) and (IV) and the heterotic-type IIB string dualities of them are in [17].

One aim in examining heterotic-type IIB string dualities in [17] is to show an interplay of perturbative gauge field and non-perturbative one by using monodromies and discriminants

• For (III) representation with J_1 identified with the dilaton [5],
the gauge symmetry in heterotic string side comes from
(I of $\mathrm{K3}_2$)×($\mathrm{U}(1)^{\otimes \Delta n_T}$ of tensor) (A_2 of T^2 cmpt.) ← strong
× (ADE sing. of $\mathrm{K3}_1$) ← weak

• For (IV) representation with t_2 identified with the dilaton,
the gauge symmetry in the heterotic string side comes from
(I of the generic $\mathrm{K3}_2$) × ($\mathrm{U}(1)^{\otimes \Delta n_T}$ of tensor) × (A_2 of T^2 cmpt.) ← weak
× (remains of ADE in $\mathrm{K3}_1$) ← strong
(If we take t_1 as the dilaton, then the remains of ADE singularity appear in the weak coupling region of b_1.)

The higher derivative couplings of vector multiplets X to the Weyl multiplet W of conformal N=2 supergravity can be expressed as a power series: $K(X, W^2) = \Sigma_{g=0}^{\infty} K_g(X)(W^2)^g$. Suppose that a holomorphic prepotential of genus zero in heterotic string of D=4 side is given by tree and one-loop contri-

[4]The definitions of the models are in the next section.

[5]There are two parameters, b_1 and b_2 due to double K3 fibrations with $b_i = e^{-2\pi t_i}$, $c_2 \cdot J_i = 24$.(i=1,2). The explanations of t_i and J_i are in section 3. If we can identify a parameter in the discriminants in IIB side with t_i in type IIA side by mirror map then $b_i \to 0$ is a weak coupling region and $b_i = 1$ will be a strong coupling region.

butions. $K_H^{(g=0)} = S(TU - \Sigma_i C^i C^i) + \mathcal{K}_H^{(1)}(T, U, C) + \mathcal{K}_H^{(NP)}(e^{-2\pi S}, T, U, C, C', V) \in \mathcal{K}_H^{(0)} + \mathcal{K}_H^{(1)}$. T and U are two Abelian vector multiplets that contain the Kaluza-Klein gauge bosons of the torus and the corresponding toroidal moduli. The scalars C^i, $i = 1, \cdots$, rank (G) in a Cartan subalgebra of G are flat directions of the effective potential and at generic values in their field space the gauge group is broken to $[U(1)]^{rank(G)}$. A vector multiplt that comes from D=6 tensor multiplet and contains a candidate of dilaton is denoted by S. $\mathcal{K}_H^{(NP)}$ summarizes the space-time instanton correction to $\mathcal{K}_H$, i.e., containing sum of trilogarithmic function, and $\mathcal{K}^{(1)}$ is the dilaton independent one-loop contribution. In a phase where both T and S have candidate of dilaton i.e.,in a double K3 fibered phase, if under $S \leftrightarrow T$ exchange, $\mathcal{K}_H^{(NP)} \rightarrow P_3^{(NP)}(T, U, C, C', V)$ and $\mathcal{K}_H^{(1)} = P_3^{(1)}(T, U, C^i) + \cdots$ for $S \rightarrow \infty$ then the trace of heterotic-heterotic duality exists where P_3 is some triple couplings and independent on S. C' can be additional vector multiplets or dual tensor-vector multiplets that are of non-perturbative orgin and do not have the canonical couplings to one-side dilaton, S. We would like to seek these phenomena ocurring between two representations. [6].

We also discuss how to identify two representations. Some identifications of CY3 phases have been done by using dual polyhedra [21, 22, 23, 24]. The method given by [1, 2] is powerful to see the property of CY3s and their relations. There are several works that discuss the relation of elliptic fibered CY3s with $\mathbf{F}_0$ base and $\mathbf{F}_2$ base [3, 26, 27, 28, 29]. The investigation in Sec. 4 is based on the topological invariant calculation done by S. Hosono [30] and serves as an extension of the earlier works. The organization of this article is as follows:

1. Introduction

[6] When taking a strong coupling region limit such as one of $J_2 = 0, c_2 \cdot J = 24$ in a double K3 fibered phase in **CFPR** model then we may find a single K3 fibered phase in **HLY** model that corresponds to this situation. In **CFPR** model side, C' can not be represented by a toric divisor however C' may be a toric divisor in the single K3 fiberd phase of **HLY** model. (In general, C' can be seen in the extremal transition [18]. However, we would like to relate them to the modular forms or the characters.) We would like to derive the information for a compact form of trilogarithmic function contribution and to compare it with that on local CY3 case because (IV) model side partition function will be represented in a simple form. We would like to express sum of trilogarithmic function as [19, 20].

2. Why we compare various models ?
3. The method to identify toric representations
4. The relation among (III), (IV) and (V) models
5. Future problem

2 Why we compare various representations ?

There are several series of CY3s that arises naturally from the heterotic-type IIA string duality. The starting point is $E_8 \times E_8$ heterotic string compactified on K3$\times\mathbf{T}^2$ with $G_1 \times G_2$ bundles with instanton numbers (k_1, k_2) such that $k_1 + k_2 = 24$ [8, 9] [7]. Using the index theorem and anomaly cancellation condition, we find spectra of D=4 N=2 heterotic string vacua, which are related to the Hodge numbers of CY3-folds in typeIIA string side with the same spectra. We list four series that have dual, type IIA string on CY3s [8, 9][8] (CY3s used in (I) and (II) series are in tables 1,2 and 3.)

1. In the first series,
   $E_8 \rightarrow G_1 = I$
   $E_8 \rightarrow G_2 = \{I, A_1, A_2, D_4, E_6, E_7, E_8\}$ that depends on n^0.
   We call this series as (I) (terminal case). (see table 2)

2. The second series,
   $E_8 \rightarrow G_1 \neq I, i.e., G_1 = \{A_1, A_2, A_3, \cdots\}$
   $E_8 \rightarrow G_2 = \{I, A_1, A_2, D_4, E_6, E_7, E_8\}$ that depends on n^0.
   We call this series as (II).

We follow the notations, such as (I), (II), etc. given in in those of [17, 29].

Most of CY3s in (I) and (II) can be extended to be CY3s with extra blow ups by adding appropriate toric points [9, 22]. ((III) and (IV) are in tables 4

[7] $(k_1 = 12 + n^0, \quad k_2 = 12 - n^0)$ where n^0 is introduced for convenience. G_1 and G_2 come from each E_8.

[8] Furthermore, there are three versions of these series by changing the type of elliptic fiber. A-chain version is in the [9, 12], where the elliptic fiber of the CY3 is $\mathbf{P}(1, 2, 3)[6]$. The extension to B or C versions with elliptic fiber $\mathbf{P}(1, 1, 2)[4]$ or $\mathbf{P}(1, 1, 1)[3]$ are also possible.

and 5) They are classified to third (i.e., an kind of $(\mathrm{I})^\dagger$) or forth series (i.e., an kind of $(\mathrm{II})^\dagger$ [9]). (III) of **CFPR** model is a similar extension from (I) [10] and (XI) of **CFPR** model from (II). We see that $(\mathrm{I}^\dagger)$ and (III) are equivalent [11].

There is a quite different type extension of CY3s which is given by [31]. Keeping the K3 fibration with the same weight, a weight of base $P^1(1,s)$ is changing in this representation of CY3s. We call them (IV) and (VI) of **HLY** model, which may relate to be $(\mathrm{I})^\dagger$ and $(\mathrm{II})^\dagger$ of **CFPR** model [12].
There is another type of the extension that has a triple K3 fibration at most [27]. We call this (V) of **LSTY** model.

3. The third series,
$E_8 \rightarrow G_1 = I$
$E_8 \rightarrow G_2 = \{I, A_1, A_2, D_4, E_6, E_7, E_8\} \times U(1)^{\otimes \Delta n_T}$ due to additional tensor multiplets. We call this series as $(\mathrm{I})^\dagger$ or (III) of **CFPR** model. (V) of **LSTY** model are also in this series. (IV) of **HLY** model may relate or belong to this series. (see table 3 and 4)

4. The forth series,
$E_8 \rightarrow G_1 \neq I, G_1 = \{A_1, A_2, A_3, \cdots\}$
$E_8 \rightarrow G_2 = \{I, A_1, A_2, D_4, E_6, E_7, E_8\} \times U(1)^{\otimes \Delta n_T}$. $(\mathrm{II})^\dagger$ or (XI), extensions from **CFPR** model are in this series. (VI) of **HLY** model may relate or belong to this series. (see table 7)

[9] $(\mathrm{I})^\dagger/(\mathrm{II})^\dagger$ means the modified (I)/(II) in [10] with extra tensor multiplets.

[10] The difference between the dual polyhedron of $(\mathrm{I})^\dagger$ and (III) is as follows [10, 15, 12]. The dual polyhedra of case $(\mathrm{I})^\dagger$ have the modified dual polyhedra of K3 part. For the case (III) in the A series, the dual polyhedron of K3 part is not modified. The highest point in the additional points is represented by the weights of the K3 part of the terminal A series in this base. One point such as $(0,*,*,*)$ is also represented by the part of the weight of the terminal K3 part. The following element in SL(4,**Z**) can transform these polyhedra into the dual polyhedron given by [15]. $\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & -1 \end{pmatrix}$. In the base of [15], the additional points make a line with $x_4 = -1$.

[11] Their triangulations coincide by identifying the replaced vertices. Furthermore, U(1) charges defined in appendix 1 for (III) of [15] matches with the that for $(\mathrm{I})^\dagger$ of [12].

[12] The relation of toric realizations of $(\mathrm{II})^\dagger$ and (XI) is similar to that of $(\mathrm{I})^\dagger$ and (III). They will be coincide with each other.

Each CY3 can be realized by a hypersurface in a toric variety. A dual polyhedron of them contains some sub dual polyhedra of K3 part, which can be K3 fibrations in some phases. Base surfaces under the elliptic fibration of these CY3s are blow-ups of the ath Hirzebruch surface, $\mathrm{Bl}(\mathbf{F})_a$. We give some explanations of models and list four dual polyhedra with $(h^{1,1}, h^{2,1}) = (5, 185)$ that we deal with in this paper.

(**I** and **II**) `Models of Aldazabal, Ibanez, Font, Quevedo and Uranga` [8, 9] (**AIFQU** model)

A dual polyhedron of this representation in [10] contains two sub dual polyhedra of K3 part as $(0, *, *, *)$ or $(*, 0, *, *)$. One of which varies according to the instanton numbers. CY3s have a $\mathbf{P}^1(1,1)$ base under this K3 fibrations, (see table 1). CY3s have a F_{n_0} base under a elltiptic fiber.

($\mathbf{I}^\dagger$) `Models of Candelas, Font, Perevalov and Rajesh` [10, 12] (**CFPR** model) a dual polyhedron in [12]of **CFPR** model

$$
\begin{array}{lrcl}
 & & & \text{difference from (III)} \\
(0) & (\ \ 0,\ \ 0,\ \ 0,\ \ 0) & & \\
(1) & (\ \ 0,\ \ 0,-1,\ \ 0) & & \\
(2) & (\ \ 1,\ \ 0,\ \ 2,\ \ 3) & & \\
(3) & (\ \ 1,\ \ 2,\ \ 2,\ \ 3) & \longleftarrow & (\ \ 1,\ \ 2,\ \ 6,\ \ 9)^{(\mathrm{III})} \\
(4) & (\ \ 0,\ \ 0,\ \ 0,-1) & & \\
(5) & (-1,\ \ 0,\ \ 2,\ \ 3) & & \\
(6) & (\ \ 0,-1,\ \ 2,\ \ 3) & \longleftarrow & (\ \ 0,-1,\ \ 0,\ \ 0)^{(\mathrm{III})} \\
(7) & (\ \ 0,\ \ 1,\ \ 2,\ \ 3) & \longleftarrow & (\ \ 0,\ \ 1,\ \ 4,\ \ 6)^{(\mathrm{III})} \\
(8) & (\ \ 1,\ \ 1,\ \ 2,\ \ 3) & \longleftarrow & (\ \ 1,\ \ 1,\ \ 4,\ \ 6)^{(\mathrm{III})} \\
(9) & (\ \ 0,\ \ 0,\ \ 2,\ \ 3)\ . & &
\end{array}
$$

($\mathrm{I}^\dagger$) have extra one cones of (1)(1,1,2,3) and (8)(1,1,2,3) in addition to those of (I). K3 part is the same as those in (I) and (III).

(**III**) `Models of Candelas, Perevalov and Rajesh` [15] (**CFPR** model)

a dual polyhedron in [15] of **CFPR** model (We use the right hand side in this paper.)

$$
\begin{array}{cccccccccccc}
 & & & & & \text{SL}(4;\mathbf{Z})\text{ trans.} & & & & \\
(0) & (& 0, & 0, & 0, & 0) \rightarrow (& 0, & 0, & 0, & 0), \\
(1) & (& 0, & 0, & 1, & 2) \rightarrow (& 0, & 0, & -1, & 0), \\
(2) & (& 1, & 0, & 0, & -1) \rightarrow (& 1, & 0, & 2, & 3), \\
(3) & (& 1, & 2, & 0, & -1) \rightarrow (& 1, & 2, & 6, & 9), \\
(4) & (& 0, & 0, & -1, & -1) \rightarrow (& 0, & 0, & 0, & -1), \\
(5) & (& -1, & 0, & 2, & -1) \rightarrow (& -1, & 0, & 2, & 3), \\
(6) & (& 0, & -1, & 1, & -1) \rightarrow (& 0, & -1, & 0, & 0), \\
(7) & (& 0, & 1, & 1, & -1) \rightarrow (& 0, & 1, & 4, & 6), \\
(8) & (& 1, & 1, & 0, & -1) \rightarrow (& 1, & 1, & 4, & 6), \\
(9) & (& 0, & 0, & 1, & -1) \rightarrow (& 0, & 0, & 2, & 3).
\end{array}
$$

A dual polyhedron contains two sub dual polyhedra of K3 parts: $(0,*,*,*)$ and $(*,0,*,*,*)$. One of which varies according to the number of tensor multiplets. CY3s have a $\mathbf{P}^1(1,1)$ base under this K3 fibrations. The latter K3 part is always $\mathbf{P}^3(1,1,4,6)[12]$. CY3s contain some double K3 fibration phases. The $h^{1,1}=5$ case has $\Delta n_T=2$ and $\mathbf{P}^3(1,1,4,6)[12]$ as both K3 fibrations.

(**IV**) `Models of Hosono, Lian and Yau` [31] (**HLY** model)

CY3s are realized by the weighted projective hypersurfaces $\mathbf{P}^4(1,s,s+1,4s+4,6s+6)[12s+12]$ with $\{s=1,2,3,4,6,8,12\}$. CY3s have single K3 fibration phase with fiber $\mathbf{P}^3(1,1,4,6)[12]$ as $(0,*,*,*)$ and base, $\mathbf{P}(1,s)$. $h^{1,1}=5$ case has $s=\Delta n_T=2$.

Using toric data, we can examine the heterotic-type IIA duality. The heterotic-type IIA string duality for (III) and (I†) has been made clear in [12, 21], which we review at first. The differences between the Hodge numbers of (I) and (III) or (I†) with $k_1+k_2+\Delta n_T=24$, $k_1=12+n^0-\Delta n_T$, $n^0=\Delta n_T$ are given by

$$
\begin{aligned}
\Delta h^{2,1} &= -h^{2,1}\,|_{\text{in (I)}} + h^{2,1}\,|_{\text{in (III)}} = -29\Delta h^{1,1}, \\
\Delta h^{1,1} &= -h^{1,1}\,|_{\text{in (I)}} + h^{1,1}\,|_{\text{in (III)}} = \Delta n_T.
\end{aligned}
$$

a dual polyhedron in [31] of **HLY** model

$$
\begin{array}{llcl}
 & & & \text{difference from (III)} \\
(0) & (\ \ 0, \ \ 0, \ \ 0, \ \ 0) & & \\
(1) & (\ \ 0, \ \ 0, \ \ 0, -1) & & \\
(2) & (\ \ 0, \ \ 0, -1, \ \ 0) & & \\
(3) & (\ \ 0, -1, \ \ 0, \ \ 0) & & \\
(4) & (-1, \ \ 0, \ \ 0, \ \ 0) & \longleftarrow & (-1, \ \ 0, \ \ 2, \ \ 3)^{(\mathrm{III})} \\
(5) & (\ \ 2, \ \ 3, \ 12, \ 18) & \longleftarrow & (\ \ 1, \ \ 0, \ \ 2, \ \ 3)^{(\mathrm{III})} \\
(6) & (\ \ 1, \ \ 2, \ \ 8, \ 12) & \longleftarrow & (\ \ 1, \ \ 2, \ \ 6, \ \ 9)^{(\mathrm{III})} \\
(7) & (\ \ 0, \ \ 1, \ \ 4, \ \ 6) & & \\
(8) & (\ \ 1, \ \ 1, \ \ 6, \ \ 9) & \longleftarrow & (\ \ 1, \ \ 1, \ \ 4, \ \ 6)^{(\mathrm{III})} \\
(9) & (\ \ 0, \ \ 0, \ \ 2, \ \ 3) & &
\end{array}
$$

The number of the tensor multiplets n_T in (IV) is given by $n_T = h^{1,1}\,(\mathrm{Bl}(\mathbf{F}_2)) - 1 = d_1 - 2d_0 - 1 = s+1,\ (s \geq 2)$, where d_i denotes the number of i-dimensional cones of the fan that describes the base $\mathrm{Bl}(\mathbf{F}_2)$. The Hodge numbers and n_T in (IV) and (V) coincide with those in (III). It seems that there exists a heterotic string on K3$\times\mathbf{T}^2$ that is dual to both the type IIA string compactified on CY3 of (IV) and that on a CY3 of (V).

Another candidate of single K3 fibered CY3 representation is

(VI) `Models of Hosono, Lian and Yau` [31] (**HLY** model)

$\mathbf{P}^2(1,s)$ based $\mathbf{P}^3(1,1,3,5)[10]$ fibered CY3s, $\mathbf{P}^4(1,s,s+1,3s+3,5s+5)[10s+10]$ with $\{s=1,2,3,5,7,9,10\}$. This representation relates to the forth series of heterotic type IIA string duality with $G_1 = A_1$ in A series. Some of CY3s satisfy the following anomaly free conditions for $s = \{2,5,7,9\}$[17] [13]. As the result of the comparison of (VI) and (II) in A series with $G_1 = A_1$, we obtain

$$\Delta h^{2,1} = h^{2,1}\,|_{(\mathrm{VI})} - h^{2,1}\,|_{(\mathrm{II})},\ \Delta h^{1,1} = h^{1,1}\,|_{(\mathrm{VI})} - h^{1,1}\,|_{(\mathrm{II})} = \Delta n_T,$$

$$h^{2,1}|_{(\mathrm{II})\ \mathrm{with} A_1} = h^{2,1}|_{(\mathrm{I})} - (12n+29),\ \Delta h^{2,1} = -(29-12)\Delta h^{1,1} = -17\Delta h^{1,1}.$$

[13] $n^0 = 0$ case with $\mathbf{F}_0$ based CY3 and $n^0 = 2$ case with $\mathbf{F}_2$ based CY3 in (II) have the different Hodge numbers. Therefore, for some CY3s with small s, the anomaly free-conditions are changed.

with $k_1 + k_2 + \Delta n_T = 24, k_1 = 12 + n^0 - \Delta n_T,\ n^0 = \Delta n_T$.
For $K3 \times T^2$ side, $12n+29$ is calculated by the index theorem and denotes the number of G_1 charged hyper multiplet fields[22, 9]. We substitute $n = n^0 - \Delta n_T$ for the extra tensor multiplets case instead $n = n^0$. Similar extensions to other $\mathrm{G}_1 \neq I$ gauge group in A chain, B and C chain versions also seem to be possible [17]. We also suppose the existence of the double K3 fibered CY3s, which is denoted as (IX). They can be obtained by the extension from A series with $G_1 = A_1$ in model (II).

(**V**) `Model of Louis et al.` [27] (**LSTY** model)
a dual polyhedron in [27] of **LSTY** model (It can be obtained by the modifications of (III) or ($\mathrm{I}^\dagger$))

$$
\begin{array}{llll}
(0) & (\ \ 0,\ \ 0,\ \ 0,\ \ 0) & & \\
(1) & (\ \ 1,\ \ 1,\ \ 2,\ \ 3) & \longleftarrow & (\ \ 1,\ \ 1,\ \ 4,\ \ 6)^{(\mathrm{III})} \\
(2) & (\ \ 1,\ \ 0,\ \ 2,\ \ 3) & & \\
(3) & (\ \ 0,-1,\ \ 2,\ \ 3) & \longleftarrow & (\ \ 0,-1,\ \ 0,\ \ 0)^{(\mathrm{III})} \\
(4) & (-1,-1,\ \ 2,\ \ 3) & \longleftarrow & (\ \ 1,\ \ 2,\ \ 6,\ \ 9)^{(\mathrm{III})} \\
(5) & (-1,\ \ 0,\ \ 2,\ \ 3) & & \\
(6) & (\ \ 0,\ \ 1,\ \ 2,\ \ 3) & \longleftarrow & (\ \ 0,\ \ 1,\ \ 4,\ \ 6)^{(\mathrm{III})} \\
(7) & (\ \ 0,\ \ 0,\ \ 2,\ \ 3) & \longleftarrow & \text{the point absent in “17” [27, p.20]} \\
(8) & (\ \ 0,\ \ 0,-1,\ \ 0) & & \\
(9) & (\ \ 0,\ \ 0,\ \ 0,-1) & &
\end{array}
$$

A toric representation of (V) with $h^{1,1}=5$ contains three dual sub polyhedra of K3 part. All K3 part are realized by $\mathbf{P}^3(1,1,4,6)$[12]. [14] By these toric data, we can see the structure of the Kähler moduli spaces CY3[15]. Given a singular ambient space, we have in general many phases in the associated Kähler moduli space of the nonsingular ambient space.

[14] The dual polyhedron of (V) coincides with that of ($\mathrm{I}^\dagger$) except one vertex: $(1,2,2,3)^{(\mathrm{III})} \rightarrow (-1,-1,2,3)^{(\mathrm{V})}$. By this, the existence of three symmetric K3 sub dual polyhedra can be seen apparently: $\{(5)(-1,0,2,3),\ (*,0,*,*),(2)(1,0,2,3)\} \leftrightarrow \{(3)(0,-1,2,3),\ (0,*,*,*),(6)(0,1,2,3)\} \leftrightarrow \{(4)(-1,-1,2,3),\ (*,*,*,*),(1)(1,1,2,3)\}$.

[15] We follow the result, notations and definitions of [1, 2].

3 The method to identify toric representation

We discuss the case when two dual polyhedra have no twisted sectors (= non-toric degree of freedom). The method that we use is given by [1, 2], that is, to derive Gromov-Witten invariants, to compare them directly and to examine the relations of CY3 phases. It is the most effective and rigorous way. Especially, if some Gromov-Witten invariants of CY3s are those of Del Pezzo surfaces, the comparison is very easy. By fixing U(1) charges, Q such as in appendix 1, we first have to calculate Mori-cones[33] and Kähler cones in each phase[16]. Mori-cones generated by the holomorphic curves $\{\ell_j\}$ are the dual of Kähler cones generated by $\{J_i\}$ [17] [18] [20]. The Gromov-Witten invariants $N(\{n_i\})$ are defiend by the instanton corrected Yukawa coupling K_{x_i,x_j,x_k}. $N(\{n_i\})$ is the instanton number of the rational curves C of multidegree $\{n_i = \int_C J_i\}$. The algebraic coordinates $\{x_i\}$ and the special coordinates $\{t_i\}$ are related to the mirror map via Mori-vectors [1, 2].

$$
\begin{aligned}
K_{t_i,t_j,t_k}(t)^{g=0}_{\mathrm{II}} &= \frac{1}{w_0(x(t))^2}\sum_{lmn}\frac{\partial x_l}{\partial t_i}\frac{\partial x_m}{\partial t_j}\frac{\partial x_n}{\partial t_k}\mathcal{K}_{x_l,x_m,x_n}(x(t)),\\
&= \mathcal{K}^0_{ijk} + \sum_{\{n_l\}} N(\{n_l\})n_in_jn_k\,\frac{\prod_l q_l^{n_l}}{1-\prod_l q_l^{n_l}},
\end{aligned}
$$

where $q_i = \mathrm{e}^{-2\pi t_i}$. Integrating back yields a trilogarithmic function. $\mathcal{K}^0_{ijk}$ is the classical part of the Yukawa couplings [21]. The dual polyhedra that we

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[16]They satisfy $\{Q_1,\cdots,Q_9\} = \{J_1,\cdots,J_5\}\cdot\begin{pmatrix}\ell_1\\ \ell_2\\ \ell_3\\ \ell_4\\ \ell_5\end{pmatrix}$. By matrix notation, J_j and Q_i are column vectors and ℓ_i are row vectors. $i = 1,\cdots 5 for h^{1,1} = 5$ case. 9 denotes the number of the points in the dual polyhedra of a CY3-fold.

[17]The complexified Kähler class J is givem by $J = \sum_{i=1}^{h^{1,1}} t_iJ_i \in H^2(M;\mathbf{C})$.

[18]Both ℓ_j and J_i are represented by the common dual basis in each models, m_i and D_i such as $J_j = \Sigma D_iA_{ij}$ [19] and $\ell_i = \Sigma S_{ij}m_j$. D_i are toric divisors corresponding Q_i charges. $D_i\cdot m_j = \delta_{ij}$. A_{ij} and S_{ij} are 5×5 transformation matrices.

[20]We can see that the volume of the curve $\ell = \sum_j n_j\ell_j$ measured by J is $\mathrm{vol}_{J(\ell)} = \sum_{i=1}^{h^{1,1}} n_it_i$.

[21]$\mathcal{K}^0_{ijk} := d_{ijk}t_it_jt_k,\quad d_{ijk} = \int_{CY3} J_i\wedge J_j\wedge J_k$. To get the Yukawa coupling from this notation, some additional normalizations factors are necessary: $t_i^3 \to \frac{1}{3!}t_i^3$ and $t_i^2 \to \frac{1}{2!}t_i^3$.

compare do not coincide with each other by SL(4,$\mathbf{Z}$) transformation. Nevertheless, in some phases, topological invariants happen to match. Note that Mori vectors do not match even if they are equivalent CY3 phases between the different models. We use the following theorem and the sub steps. Wall's Theorem says that the agreements of classical invariants, $c_2 \cdot J$ and $\mathcal{K}^0$, lead to the agreements of topology as well as Gromov–Witten invariants [32] in two CY3s. We can narrow down candidates of the transformation matrix by comparing values of $c_2 \cdot J_i$.

- criterion 1 :
  If $c_2 \cdot J$ and $\mathcal{K}^0$ match then it leads to the agreement of the Gromov–Witten invariants, $N(\{n_i\})$. In this case, the number of the K3 fibrations in two phases is the same [22].

- criterion 2 :
  To compare the two phases (regardless of their jurisdiction, i.e., the different models or the same models), we can use a candidate of transformation matrix of topological invariants by combining some transformation matrices of divisors. We can make such a matrix by replacing a divisor of the original phase by another divisor. These divisors have the same $c_2 \cdot J_i$ and d_{ijk}. If this matrix is integer valued and transforms topological invariants, then these two phases are the same ; Let (J_i, ℓ_j) and $({J'}_i, {\ell'}_j)$ be some generators of the Kähler and the Mori cones of the two equivalent CY3 phases. An integer-valued matrix of divisors such as ${J'}_i^{(B)} = \sum_{j=1}^{h^{1,1}} J_j^{(A)} (M_{(AB)})_{j,i}, \quad {\ell'}_i^{(B)} = \sum_{j=1}^{h^{1,1}} (M_{(AB)})^{T(-1)}{}_{i,j} \ell_j^{(A)}$, transform topological invariants, $c_2 \cdot J'_j = \sum_{j=1}^{h^{1,1}} M_{(\mathrm{AB})ij} c_2 \cdot J'_i$, $d'_{ijk} = \int_{\mathrm{CY3}} J'_i \wedge J'_j \wedge J'_k = \sum_{lmn} M_{(\mathrm{AB})il} M_{(\mathrm{AB})jm} M_{(\mathrm{AB})kn} \int_{\mathrm{CY3}} J_l \wedge J_m \wedge J_n$, and $N(\{n_i^{(A)}\}) = N'(\{{n'}_i^{(B)}\}), \quad \text{for} \quad {n'}_i^{(B)} = \sum_{j=1}^{h^{1,1}} n_j^{(A)} (M_{(\mathrm{AB})})_{j,i}$ [23] [24] [25].

[22] In these cases, we can see some mappings of the ambient space data between two equivalent whole Kähler cones of two models.

[23] A curve $[C]$ admits the expansion $[C] = \sum_{i=1}^{h^{1,1}} n_i \ell_i = \sum_{j=1}^{h^{1,1}} {n'}_j {\ell'}_j$.

[24] A transformation matrix, M can contain some negative integers even if they are in the same phase. However,$\{n_i\}$ and $\{n'_i\}$ are bijective and should contain only positive integers.

[25] A transformation matrix of whole Kähler cones such as $M^{(\mathrm{IV}),\mathrm{T}}_{(\mathrm{AB})} = A^{(\mathrm{IV}),\mathrm{T}}_{(\mathrm{B})} A^{(\mathrm{IV}),\mathrm{T}-1}_{(\mathrm{A})}$

4 The relation among (III), (IV) and (V) models

There are two characteristic points about the triangulations of (III), (IV) and (V) with $(h^{1,1},h^{2,1})=(5,185)$.

The first point is about the feature of the Gromov-Witten invariants. All phases of them contain Del Pezzo 4-cycles, B_8 [26] in many ways[27]. The existence of B_8 in $\mathbf{F}_1$ based elliptic fibered CY3 with $(h^{1,1},h^{2,1})=(4,214)$ has already been investigated [6]. $(h^{1,1},h^{2,1})=(5,185)$ case has one more extra blow up point than $(h^{1,1},h^{2,1})=(4,214)$ case.

The second point is that the different triangulations of them do not lead to the different phase. In general, a CY3 phase is specified by a particular triangulation of the polyhedron. However, some different triangulations (called phases in this article) do not lead to different CY3 phase [11]. In that case, the conclusion is that the singularities on the submanifolds blown up to specify each phase do not contribute to CY3 phases. This property depends on the dimensions of the submanifold that contains them [28]. Some triangulations in (III), (IV) and (V) resulted into this case.

There are five phases in **HLY** model, eight phases in **CFPR** model and eighteen phases in **LSTY** model which are specified by the triangulation (see table 7). The identifications of CY3 phases by the criterion 1 is shown in table 9. The phases in the same line in table 9 have the same topological invariants. Four phases of a single K3 fibration in **HLY** model can be identified with four phases of a single K3 fibration in **CFPR** model. 15 phases of (V) in **CFPR**

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in the same model can not transform topological invariants, because they depend on the ambient space data specified by the triangulation. (A and B denote phase names for example.) However, modifying or combining A, we can get a transformation matrix of equivalent phases. Though in one model, all the divisors can not always be represented by the data of the ambient space, they will be transformed or related to those of the another model. We confirmed this by including (V).

[26] B_8 is given by $E_8=\mathbf{P}^2(1,2,3)[6]$ fibered 4-cycle and has eight blow up points. The property of Gromov-Witten invariants in B_8 is ruled by this elliptic fiber.

[27] Most phases have B_6 in six ways. For example, we can reduce Mori vectors of CY3 in triple K3 fibration phase, α_{10} to those of Del Pezzo B_8 in six ways.

[28] In this case, the dimension of the submanifold is codim 2 of CY3 +1.

model can be identified with four phases of (III) of **CFPR** model.

By using criterion 2, we can identify one pair of phases such as the phase A with a single fibration of **HLY** model and the phase g with a double K3 fibration of **CFPR** model. We can also identify the left four phases of (III) in **CFPR** model including phases with double K3 fibration and the left three phases in (V) of **LSTY** model including a triple K3 fibration case with the phases in (IV) of **HLY** model. The result is in table 10.

In conclusion, there are only five topologically nonequivalent phases defined by the different triangulation in (IV) of **HLY** model for $\Delta n_T = 2$. The other phases in (III) of **CFPR** model and (V) of **LSTY** model are equivalent to these five phases. Each model that contains topologically equivalent phases is a local coordinate representation of the same CY3-fold [29].

5 Future problem

In this paper, we derived the relationship of three CY3 models with $(h^{1,1}h^{2,1})=(5,185)$. We come to the conclusion that three models and their extensions satisfy N=2 and N=1 string dualities[35] because they are all local representations of the same CY3. This is the starting point of the comparison of (III) of **CFPR** model and (IV) of **HLY** model to derive the example of N=2 and N=1 string dualities on two toric representations of the same CY3 model. In $h^{1,1}=5$ case, both K3 fibrations are the same K3 therefore interplay of non-perturbative and perturbative gauge fields, i.e., the trace of heterotic-heterotic string duality is not seen [27][30]. However, this CY3 is an extension of phase 6

[29] All phases in (IV) are represented by the simplicial cones. However, half of (III) phases and most of (V) phases are not simplicial cones. It is difficult to take five true phases by taking the union of Kähler cones of the equivalent phases among (IV), (III) and (V). Because, we must get all virtual Mori-vector of the one model side, which corresponding to Mori-vectors of the equivalent phase of the other model to take the intersection. For example, we can not get the cap of Mori-vectors of phase A and g and α_{10} in (III), (IV) or (V) side .

[30] For triple K3 fibered phase of α_{10}= phase 17 of [27], three t_i with $C_2 \cdot J_i = 24$ are symmetric in $\mathcal{K}^0$ therefore, the change $S \leftrightarrow T$ is symmetric. The IIA string side topological part of prepotential is $\mathcal{K}^0 = STU - \frac{1}{2}UV_Y^2 - \frac{1}{2}UW_Y + \frac{1}{4}U^3$ by changing t_i into heterotic side variables. V_Y and W_Y are two tensor multiplets [27].

with (4,214) in [27] which is related to the non-critical string model[34] though the part of CY3 where B_8 exists is different. We conjecure that the relation of **HLY** and **CFPR** models with higher Hodge numbers may also be interpreted by shrinking B_8 and flops on ($\mathbf{P}^1(1,1)$ based $\mathbf{P}^1(1,s)$ fibered) Hirzebruch surface based elliptic fiberd CY3-fold [31]. For higher Hodge numbers case, the type of double K3 fibration is different and tree level topological three-point function with a dilaton, T should come from non-perturbative terms of with another dilaton, S side in the double K3 fibration phase. Therefore, the trace of heterotic-heterotic duality will be seen apparently.

4D N=2 super YM theory can also be analyzed as the heterotic strings compactified on $K3\times\mathbf{T}^2$ in the weak coupling region. The threshold correction of case (I) in heterotic string side has been given by the calculation of the partition function [39, 40, 41, 42]. By combining their method with generalized modular forms [43] and the result of local B_8 string model [34, 44], we will be able to derive the perturbative Yukawa couplings for the extra tensor multiplets case by comparing the Gromov-Witten invariant data of type II [32] [33].

Furthermore, we would like to compare the partition function on global CY3 and on local CY3 [36] [34]. By identification of a Mori-vector with one from elliptic fiber i.e., and by taking the limit of large elliptic fiber in GKZ equations as $z_f \to 0$, we can get a local CY3-fold [37]. We would like to use this data to compactify type II/M and to get 5-dim. gauge theory on $M_4 \times S^1$

[31] Type IIB string side on (IV) case is a theory on $\mathbf{P}^1(1,s)$ based ALE fibered one, which might correspond to the non-perturbative property of (III).

[32] Take for a simple example, phase 6 of (4,214) case in[27], Mori-vectors of CY3 reduce to those of Del Pezzo B_8 surfaces in three ways, which agrees with table 8. By decomposing the result of [44] and replacing some part to match with brown up F_2 data in table 3 of [37], we will be able to derive the partition function. (See appendix 6 and [45]).

[33] An examination of relations among $h^{1,1}=4$ and $h^{1,1}=5$ models and their partition functions in type II-heterotic string web in table [27, 28] will be interesting by applying a method in this paper. The phase 5 given by in [27] has non-symmetrical double K3 fibrations and the trace of heterotic-heterotic string duality, which is contained in CFPR model with (8,164).

[34] For example, the base of CY3-fold of phase 6 with (4,214) is F_2 with one point blown up and the same as those of CY3-folds [38] up to SL(2,Z) transformation. Base of (V) model is also the same as those of them.

[38][35].

Acknowledgment
I would like to thank K. Mohri for fruitful discussions. I would like to thank S. Hosono for his help in calculating Mori vectors and topological invariants in this paper including the method of identification of phases. I would like to thank to N. Sakai for useful discussions.

Appendix 1 Linear relations among the vertices

We list some linear relations of one-cones in the dual polyhedra, $Q_{(i)}$. They are called U(1) charges and have SL(5,Z) option. (We use charges in the left hand side.)

- transposed U(1) in (III)[36] [37]

[35] Work is in progress.

[36] In criterion I cases of these models, we can choose U(1) charges so that divisors with the same topological invariants in two models have the same transformation matrices from the common basis of Kähler cones. Namely, for fixed Mori-vectors, we choose U(1) charges of two models so that at least $h^{1,1}$ numbers of divisors (i.e.,linear combinations of basis of Kähler cones) with same topological invariants have common values in two models. For example, in phase c and B by using $U(1)^{X_1}$, we can represent the corresponding divisor in the same value of the transformation matrices, (see appendix 2). We can use this identification of divisors to seek a corresponding divisor from one model to another model and to derive matrices in criterion 2. The mappings of U(1) charges to identify two models are not unique and we must classify phases according to the correspondence of divisor representation of the same topological invariants in two models and decide which mapping we should use to compare by case by case.

[37] $$X_1 : (Q_i)^{(\mathrm{III})} \to (Q_i)^{(\mathrm{III})}_{X_1} = \begin{pmatrix} 2Q^{(\mathrm{III})}_{2i} + Q^{(\mathrm{III})}_{1i} \\ Q^{(\mathrm{III})}_{2i} + Q^{(\mathrm{III})}_{1i} \\ Q^{(\mathrm{III})}_{3i} \\ Q^{(\mathrm{III})}_{4i} + Q^{(\mathrm{III})}_{2i} \\ Q^{(\mathrm{III})}_{5i} \end{pmatrix} \quad \text{and} \quad (Q_j)^{(\mathrm{IV})} = \begin{pmatrix} Q_{1j} \\ Q_{2j} \\ Q_{3j} \\ Q_{4j} \\ Q_{5j} \end{pmatrix}^{(\mathrm{IV})},$$

:

$$\begin{array}{lcl}
(Q_1,\ Q_2,\ Q_3,Q_4,Q_5,Q_6,Q_7,Q_8,Q_9) & & (Q_1,\ Q_2,\ Q_3,Q_4,Q_5,Q_6,Q_7,Q_8,Q_9)^{X_1}\\
(\ 4,-1,\ 1,\ 6,\ 0,\ 2,\ 0,\ 0,\ 0) & \rightarrow & (\ 12,\ 1,\ 1,18,\ 2,\ 2,\ 0,\ 0,\ 0)\\
(\ 4,\ 1,\ 0,\ 6,\ 1,\ 0,\ 0,\ 0,\ 0) & \rightarrow & (\ 8,\ 0,\ 1,\ 12,\ 1,\ 2,\ 0,\ 0,\ 0)\\
(\ 4,\ 0,\ 0,\ 6,\ 0,\ 1,\ 1,\ 0,\ 0) & \rightarrow & (\ 4,\ 0,\ 0,\ 6,\ 0,\ 1,\ 1,\ 0,\ 0)\\
(\ 2,-1,\ 0,\ 3,\ 0,\ 1,\ 0,\ 1,\ 0) & \rightarrow & (\ 6,\ 0,\ 0,\ 9,\ 1,\ 1,\ 0,\ 1,\ 0)\\
(\ 2,\ 0,\ 0,\ 3,\ 0,\ 0,\ 0,\ 0,\ 1) & \rightarrow & (\ 2,\ 0,\ 0,\ 3,\ 0,\ 0,\ 0,\ 0,\ 1).
\end{array}$$

• transposed U(1) in (IV) • transposed U(1) in (V) :

$$\begin{array}{ll}
(Q_1,Q_2,Q_3,Q_4,Q_5,Q_6,Q_7,Q_8,Q_9) & (Q_1,Q_2,Q_3,Q_4,Q_5,Q_6,Q_7,Q_8,Q_9)\\
(18,\ 12,\ 3,\ 2,\ 1,\ 0,\ 0,\ 0,\ 0) & (\ 1,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 4,\ 6)\\
(12,\ 8,\ 2,\ 1,\ 0,\ 1,\ 0,\ 0,\ 0) & (\ 0,\ 0,\ 1,\ 0,\ 0,\ 1,\ 0,\ 4,\ 6)\\
(\ 6,\ 4,\ 1,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0) & (\ 0,\ 1,\ 0,\ 0,\ 1,\ 0,\ 0,\ 4,\ 6)\\
(\ 9,\ 6,\ 1,\ 1,\ 0,\ 0,\ 0,\ 1,\ 0) & (\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 0,\ 6,\ 9)\\
(\ 3,\ 2,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1). & (\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 2,\ 3)
\end{array}$$

Appendix 2 Some pairs of the equivalent phases between (III) and (IV)

There are three pairs between (III) and (IV) :

• phase c= phase B:

• $\{J_1,\cdots,J_5\}^{(\mathrm{III})}_{\mathrm{c}} =$ • $\{J_1,\cdots,J_5\}^{(\mathrm{IV})}_{(\mathrm{B})} =$

$$\{D_1,\cdots,D_5\}^{(\mathrm{III})}\begin{pmatrix}2&1&0&1&0\\2&3&1&1&2\\2&2&0&1&1\\1&0&0&0&0\\1&1&0&0&0\end{pmatrix}^{(\mathrm{III})}_{(\mathrm{c})} \quad \Rightarrow\{D_1,\cdots,D_5\}^{(\mathrm{IV})}\begin{pmatrix}6&7&2&3&4\\4&4&1&2&2\\2&2&0&1&1\\3&3&1&1&2\\1&1&0&0&0\end{pmatrix}^{(\mathrm{IV})}_{(\mathrm{B})},$$

$$\{c_2\cdot J_1,\cdots,c_2\cdot J_5\}^{(\mathrm{III})}_{(\mathrm{c})}=\{72,82,24,36,48\} \quad \Rightarrow\{c_2\cdot J_1,\cdots,c_2J_5\}^{(\mathrm{IV})}_{(\mathrm{B})}=\{72,82,24,36,48\},$$

$\mathcal{K}^0_{\ \mathrm{B}} = \mathcal{K}^0_{\ (\mathrm{c})} = 6t_1^3+7t_1^2t_2+7t_1t_2^2+7t_2^3+2t_1^2t_3+2t_1t_2t_3+2t_2^2t_3+3t_1^2t_4+3t_1t_2t_4+$
$3t_2^2t_4+t_1t_3t_4+t_2t_3t_4+t_1t_4^2+t_2t_4^2+4t_1^2t_5+4t_1t_2t_5+4t_2^2t_5+t_1t_3t_5+t_2t_3t_5+$

$2t_1t_4t_5 + 2t_2t_4t_5 + 2t_1t_5^2 + 2t_2t_5^2$.

• phase d= phase C:

$\mathcal{K}^0_{(\mathrm{C})} = \mathcal{K}^0_{(\mathrm{d}\ )} = 6t_1^3 + 7t_1^2t_2 + 7t_1t_2^2 + 7t_2^3 + 8t_1^2t_3 + 8t_1t_2t_3 + 8t_2^2t_3 + 8t_1t_3^2 + 8t_2t_3^2 + 8t_3^3 + 9t_1^2t_4 + 9t_1t_2t_4 + 9t_2^2t_4 + 9t_1t_3t_4 + 9t_2t_3t_4 + 9t_3^2t_4 + 9t_1t_4^2 + 9t_2t_4^2 + 9t_3t_4^2 + 9t_4^3 + 3t_1^2t_5 + 3t_1t_2t_5 + 3t_2^2t_5 + 3t_1t_3t_5 + 3t_2t_3t_5 + 3t_3^2t_5 + 3t_1t_4t_5 + 3t_2t_4t_5 + 3t_3t_4t_5 + 3t_4^2t_5 + t_1t_5^2 + t_2t_5^2 + t_3t_5^2 + t_4t_5^2$. [38]

• phase b= phase D:

$\mathcal{K}^0_{(\mathrm{D})} = \mathcal{K}^0_{(\ \mathrm{b})} = 6t_1^3 + 8t_1^2t_2 + 8t_1t_2^2 + 8t_2^3 + 2t_1^2t_3 + 2t_1t_2t_3 + 2t_2^2t_3 + 7t_1^2t_4 + 8t_1t_2t_4 + 8t_2^2t_4 + 2t_1t_3t_4 + 2t_2t_3t_4 + 7t_1t_4^2 + 8t_2t_4^2 + 2t_3t_4^2 + 7t_4^3 + 4t_1^2t_5 + 4t_1t_2t_5 + 4t_2^2t_5 + t_1t_3t_5 + t_2t_3t_5 + 4t_1t_4t_5 + 4t_2t_4t_5 + t_3t_4t_5 + 4t_4^2t_5 + 2t_1t_5^2 + 2t_2t_5^2 + 2t_4t_5^2$.

There is another pair. • phase a = phase E [39]

• $\{J_1, \cdots, J_5\}^{(\mathrm{III})}_{(\mathrm{a})} =$ • $\{J_1, \cdots, J_5\}^{(\mathrm{IV})}_{(\mathrm{E})} =$

$$\{D_1, \cdots, D_5\}^{(\mathrm{III})} \begin{pmatrix} 2 & 3 & 2 & 4 & 1 \\ 2 & 2 & 0 & 2 & 1 \\ 2 & 2 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}^{(\mathrm{III})}_{(\mathrm{a})} \quad \Rightarrow \{D_1, \cdots, D_5\}^{(\mathrm{IV})} \begin{pmatrix} 6 & 7 & 2 & 8 & 3 \\ 4 & 3 & 1 & 5 & 2 \\ 2 & 2 & 0 & 2 & 1 \\ 3 & 3 & 1 & 3 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}^{(\mathrm{IV})}_{(\mathrm{E})},$$

$$\{c_2 \cdot J_1, \cdots, c_2 \cdot J_5\}^{(\mathrm{III})}_{(\mathrm{a})} = \{72, 82, 24, 92, 36\} \quad \Rightarrow \{c_2 \cdot J_1, \cdots, c_2 \cdot J_5\}^{(\mathrm{IV})}_{(E)} = \{72, 82, 24, 92, 36\},$$

$\mathcal{K}^0_{(\mathrm{a})} = \mathcal{K}^0_{(\mathrm{E})} = 6t_1^3 + 7t_1^2t_2 + 7t_1t_2^2 + 7t_2^3 + 2t_1^2t_3 + 2t_1t_2t_3 + 2t_2^2t_3 + 8t_1^2t_4 + 8t_1t_2t_4 + 8t_2^2t_4 + 2t_1t_3t_4 + 2t_2t_3t_4 + 8t_1t_4^2 + 8t_2t_4^2 + 2t_3t_4^2 + 8t_4^3 + 3t_1^2t_5 + 3t_1t_2t_5 + 3t_2^2t_5 +$

[38] We can see a mapping from (III) to (IV) from these three pairs. In some special cases, we might represent the corresponding divisor in the same value of the transformation matrices from the basis in criterion 1, i.e., the mapping of X_1 of U(1) charge corresponds to this mapping between two models:

$$X_1 : \begin{pmatrix} A_{1i} \\ A_{2i} \\ A_{3i} \\ A_{4i} \\ A_{5i} \end{pmatrix}^{(\mathrm{III})} \Rightarrow \begin{pmatrix} A_{2i} \\ A_{2i} \\ A_{3i} \\ A_{4i} \\ A_{5i} \end{pmatrix}^{(\mathrm{IV})}, \begin{pmatrix} A_{1i} \\ A_{2i} \\ A_{3i} \\ A_{4i} \\ A_{5i} \end{pmatrix}^{(\mathrm{IV})} = \begin{pmatrix} 2A_{2i}^{(\mathrm{III})} + A_{1i}^{(\mathrm{III})} \\ A_{2i}^{(\mathrm{III})} + A_{1i}^{(\mathrm{III})} \\ A_{3i}^{(\mathrm{III})} \\ A_{4i}^{(\mathrm{III})} + A_{2i}^{(\mathrm{III})} \\ A_{5i}^{(\mathrm{III})} \end{pmatrix}.$$

[39] This pair satifies another mapping.

$t_1t_3t_5 + t_2t_3t_5 + 3t_1t_4t_5 + 3t_2t_4t_5 + t_3t_4t_5 + 3t_4^2t_5 + t_1t_5^2 + t_2t_5^2 + t_4t_5^2$.

Appendix 3 An example of equivalent phases

Mori-vectors in phase A of (IV) and g of (III) are given by

$$
\begin{array}{cc}
\{\ell_i\}_{(\mathrm{A})} & \{\ell_i\}_{(\mathrm{g})} \\
(\ 3,\ 2,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1) & (\ 2,\ 0,\ 0,\ 3,\ 0,\ 0,\ 0,\ 0,\ 1) \\
(\ 0,\ 0,\ 0,\ 0,\ 1,-2,\ 1,\ 0,\ 0) & (\ 0,\ 0,\ 1,\ 0,\ 1,\ 0,-2,\ 0,\ 0) \\
(\ 0,\ 0,\ 0,\ 1,\ 0,\ 1,-2,\ 0,\ 0) & (\ 0,\ 1,\ 1,\ 0,\ 0,\ 0,\ 0,-2,\ 0) \\
(\ 0,\ 0,\ 1,\ 0,\ 1,\ 0,\ 0,-2,\ 0) & (\ 0,-1,\ 0,\ 0,\ 0,\ 1,\ 0,\ 1,-1) \\
(\ 0,\ 0,\ 0,\ 0,-1,\ 1,\ 0,\ 1,-1) & (\ 0,\ 0,-1,\ 0,\ 0,\ 1,\ 1,\ 1,-1)
\end{array}
$$

They satisfy $Q_i^{(A)} = A_{ij}^{(A)} \ell_j^{(A)}$ etc.

We show a transformation matrix from phase A to phase g, M_{Ag} :phase A of (IV) $\rightarrow$ phase g of (III) [40].

$$
M_{\mathrm{Ag}}^{\mathrm{T}} := A_{(g)}^{(\mathrm{IV}),\mathrm{T}} A_{(A)}^{(\mathrm{IV}),\mathrm{T}\,(-1)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}
$$

$\bullet\ \{J_1, \cdots, J_5\}_{(\mathrm{g})}^{(\mathrm{III})} =$

$\{D_1, \cdots, D_5\}^{(\mathrm{IV})} A_{(\mathrm{g})}^{(\mathrm{III})} =$

$$
\{D_1, \cdots, D_5\}^{(\mathrm{IV})} \begin{pmatrix} 2 & 0 & 1 & 2 & 0 \\ 2 & 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}_{(\mathrm{g})}^{(\mathrm{III})}
$$

$\{c_2 \cdot J_1, \cdots, c_2 \cdot J_5\}_{(\mathrm{g})}^{(\mathrm{III})} = \{72, 24, 36, 24, 48\}$

$\bullet\ \{J_1, \cdots, J_5\}_{(\mathrm{g})}^{(\mathrm{IV})} =$

$\overset{X_1}{\Rightarrow} \{D_1, \cdots, D_5\}^{(\mathrm{IV})} A_{(\mathrm{g})}^{(\mathrm{IV})} =$

$$
\overset{X_1}{\Rightarrow} \{D_1, \cdots, D_5\}^{(\mathrm{IV})} \begin{pmatrix} 6 & 2 & 3 & 2 & 4 \\ 4 & 1 & 2 & 2 & 2 \\ 2 & 0 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}_{(\mathrm{g})}^{(\mathrm{IV})},
$$

$\Rightarrow \{c_2 \cdot J_1, \cdots, c_2 \cdot J_5\}_{(g)}^{(\mathrm{IV})} = \{72, 24, 36, 24, 48\}$,

[40] We can derive the same matrix as $M_{(\mathrm{Ag})} = M_{(\mathrm{cg})}^{(\mathrm{III})} M_{(\mathrm{AB})}^{(\mathrm{IV})}$ via phase B=phase c.

We use the following identification of divisors. $\{c_2 \cdot J_i\} \ni \{72', 48\} \rightarrow \{72' - 48 = 24, 48\}$, which is transformed to a divisor with $c_2 \cdot J_i = 24$ in phase g of (III) model[41].

$$J_i^{(\mathrm{IV})}|_{c_2 \cdot J_i = 72'} - J_i^{(\mathrm{IV})}|_{c_2 \cdot J_i = 48} = J_i^{(\mathrm{IV})}|_{c_2 \cdot J_i = 24} \rightarrow J_i^{(\mathrm{III})}|_{c_2 \cdot J_i = 24}$$

$$= \{D_1, \cdots, D_5\}^{(\mathrm{IV})} \left(\begin{pmatrix} 6 \\ 4 \\ 2 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ 1 \\ 2 \\ 0 \end{pmatrix} \right)^{(\mathrm{IV})} = \{D_1, \cdots, D_5\}^{(\mathrm{IV})} \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}^{(\mathrm{IV})} \xrightarrow{X_1^{(-1)}} \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}^{(\mathrm{III})}$$

$\bullet \{J_1, \cdots, J_5\}^{(\mathrm{IV})}_{(\mathrm{A})} =$ $\bullet \{J_1, \cdots, J_5\}^{(\mathrm{IV})}_{(\mathrm{g})} =$

$$\{D_1, \cdots, D_5\}^{(\mathrm{IV})} \begin{pmatrix} 6 & 4 & 2 & 3 & 6 \\ 4 & 2 & 1 & 2 & 4 \\ 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 1 & 3 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}^{(\mathrm{IV})}_{(\mathrm{A})} \overset{M^T_{Ag} J^T_{(A)} = J^T_{(g)}}{\Rightarrow} \{D_1, \cdots, D_5\}^{(\mathrm{IV})} \begin{pmatrix} 6 & 2 & 4 & 2 & 3 \\ 4 & 1 & 2 & 2 & 2 \\ 2 & 0 & 1 & 1 & 1 \\ 3 & 1 & 2 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}^{(\mathrm{IV})}_{(\mathrm{g})},$$

$$\{c_2 \cdot J_1, \cdots, c_2 \cdot J_5\}^{(\mathrm{IV})}_{(\mathrm{A})} = \{72, 48, 24, 36, 72'\} \quad \Rightarrow \{c_2 \cdot J_1, \cdots, c_2 \cdot J_5\}^{(\mathrm{IV})}_{(g)} = \{72, 24, 48, 24, 36\},$$

This matrix is a one to one mapping of topological invariants. These phases are topologically equivalent. The Gromov-Witten invariants transform

$$\begin{array}{rclcrcl} N(1,0,0,0,1)_{(\mathrm{A})} & = & 252 & \rightarrow & N(1,0,0,1,0)_{(\mathrm{g})} & = & 252, \\ N(1,0,0,1,1)_{(\mathrm{A})} & = & 252 & \rightarrow & N(1,0,1,1,0)_{(\mathrm{g})} & = & 252, \\ N(1,1,0,1,1)_{(\mathrm{A})} & = & 252 & \rightarrow & N(1,0,1,0,1)_{(\mathrm{g})} & = & 252. \\ N(1,1,0,0,1)_{(\mathrm{A})} & = & 252 & \rightarrow & N(1,0,0,0,1)_{(\mathrm{g})} & = & 252. \\ N(1,1,1,1,1)_{(\mathrm{A})} & = & 252 & \rightarrow & N(1,1,1,0,1)_{(\mathrm{g})} & = & 252. \\ N(1,1,1,0,1)_{(\mathrm{A})} & = & 252 & \rightarrow & N(1,1,0,0,1)_{(\mathrm{g})} & = & 252. \end{array}$$

Appendix 4 The other examples of criterion 2

[41] In this paper, we omitted most data of (V) model and mappings since the procedure is the same, though it is complicated due to the redundancy of K3 fiberations.

The list of some transformation matrices of the topological invariants of the equivalent phases. The mappings of $c_2 \cdot J_i$ in the equivalent phases :

- $g \to \alpha_{10}, B \to$ e, D $\to$ f : $\{24, 48\} \to \{24, 24 = 48 - 24\}$
- $\alpha_3 \to$ f: $\{72, 82, 82\} \to \{72, 82, 92 = 82 + 82 - 72\}$
- $\alpha_{18} \to$ C : $\{72, 82, 92, 92'\} \to \{72, 82, 92, 102 = 92 + 92 - 82'\}$.
- h and E : $\{92, 82, 82'\} \to \{92, 82, 72 = 82' - 92 + 82\}$

$$M^{\mathrm{T}}_{(\mathrm{ce})} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad M^{\mathrm{T}}_{(\mathrm{bf})} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$M^{\mathrm{T}}_{(\alpha_3\mathrm{f})} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & 0 & 0 \end{pmatrix}, \quad M^{\mathrm{T}}_{(\alpha_{18}\mathrm{d})} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
$$M^{\mathrm{T}}_{(\mathrm{g}\alpha_{10})} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad M^{\mathrm{T}}_{(\text{ ha })} = \begin{pmatrix} 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

For the others, they are not equivalent [42].

Appendix 5 Topological data of $\mathcal{K}^0$ and $c_2 \cdot J$

We list the topological data of three models [43].

(III) There are 8 phases. We list the data of four phases

- phase e: $\mathcal{K}^0_{(\mathrm{e})} = 6t_1^3 + 3t_1^2 t_2 + t_1 t_2^2 + 2t_1^2 t_3 + t_1 t_2 t_3 + 2t_1^2 t_4 + t_1 t_2 t_4 + t_1 t_3 t_4 + 7t_1^2 t_5 + 3t_1 t_2 t_5 + t_2^2 t_5 + 2t_1 t_3 t_5 + t_2 t_3 t_5 + 2t_1 t_4 t_5 + t_2 t_4 t_5 + t_3 t_4 t_5 + 7t_1 t_5^2 +$

[42] • $\{A, g, \alpha_{10}\} \neq \{B, C, D, E\}$. $\{c_2 \cdot J_i\}_{A,g,\alpha_{10}}$ are only multiple of 12. $\{c_2 \cdot J_i\}_{B,C,D,E}$ are not.
- $B \neq E$, $\{c_2 \cdot J\}_B \ni \{48\}$, $\{c_2 \cdot J\}_E \not\ni \{48\}$. To make 48 from 24, $72'$ is necessary.
- $B, D, E \neq C$, $\{c_2 \cdot J\}_{B,D,E} \ni \{24\}$, $\{c_2 \cdot J\}_C \not\ni \{24\}$
- $B, E \neq D$, $\{c_2 \cdot J\}_{B,E} \ni \{36\}$, $\{c_2 \cdot J\}_D \not\ni \{36\}$.

[43] We also list the ring data in table 8. Topological invariants are calculated by the method in [2]. The author thanks S. Hosono for his help in the calculation.

$3t_2t_5^2+2t_3t_5^2+2t_4t_5^2+7t_5^3$;
$c_2\cdot J_{(\mathrm{e})}=\{72,36,24,24,82\}$;

- phase f: $\mathcal{K}^0_{(\mathrm{f})}=6t_1^3+7t_1^2t_2+7t_1t_2^2+7t_2^3+2t_1^2t_3+2t_1t_2t_3+2t_2^2t_3+2t_1^2t_4+2t_1t_2t_4+2t_2^2t_4+t_1t_3t_4+t_2t_3t_4+8t_1^2t_5+8t_1t_2t_5+8t_2^2t_5+2t_1t_3t_5+2t_2t_3t_5+2t_1t_4t_5+2t_2t_4t_5+t_3t_4t_5+8t_1t_5^2+8t_2t_5^2+2t_3t_5^2+2t_4t_5^2+8t_5^3$;
$c_2\cdot J_{(\mathrm{f})}=\{72,\ 82,\ 24,\ 24,\ 92\}$;

- phase g: $\mathcal{K}^0_{(\mathrm{g})}=6t_1^3+2t_1^2t_2+3t_1^2t_3+t_1t_2t_3+t_1t_3^2+2t_1^2t_4+t_1t_2t_4+t_1t_3t_4+4t_1^2t_5+t_1t_2t_5+2t_1t_3t_5+2t_1t_4t_5+2t_1t_5^2$;
$c_2\cdot J_{(\mathrm{g})}=\{72,\ 24,\ 36,\ 24,\ 48\}$;

- phase h: $\mathcal{K}^0_{(\mathrm{h})}=t_1^2t_3+t_1t_2t_3+3t_1t_3^2+2t_2t_3^2+7t_3^3+t_1^2t_4+t_1t_2t_4+3t_1t_3t_4+2t_2t_3t_4+8t_3^2t_4+3t_1t_4^2+2t_2t_4^2+8t_3t_4^2+8t_4^3+t_1^2t_5+t_1t_2t_5+3t_1t_3t_5+2t_2t_3t_5+8t_3^2t_5+3t_1t_4t_5+2t_2t_4t_5+8t_3t_4t_5+8t_4^2t_5+3t_1t_5^2+2t_2t_5^2+8t_3t_5^2+8t_4t_5^2+7t_5^3$;
$c_2\cdot J_{(\mathrm{h})}=\{36,\ 24,\ 82,\ 92,\ 82\}$;

(IV) There are five phases called A, B, C, D, E. We list only the phase A since the other four phase data coincide with the data of (III).

- phase A: $\mathcal{K}^0_{(\mathrm{A})}=6t_1^3+4t_1^2t_2+2t_1t_2^2+2t_1^2t_3+t_1t_2t_3+3t_1^2t_4+2t_1t_2t_4+t_1t_3t_4+t_1t_4^2+6t_1^2t_5+4t_1t_2t_5+2t_1t_3t_5+3t_1t_4t_5+6t_1t_5^2$;
$c_2\cdot J_{(\mathrm{A})}=\{72,\ 48,\ 24,\ 36,\ 72'\}$;

(V) There are 18 phases. We list only the data of phases 3, 10 and 18, since the other 15 data coincide with those in case (III).

- phase α_3: $\mathcal{K}^0_{(\alpha_3)}=7t_1^3+8t_1^2t_2+8t_1t_2^2+7t_2^3+7t_1^2t_3+8t_1t_2t_3+7t_2^2t_3+7t_1t_3^2+7t_2t_3^2+6t_3^3+2t_1^2t_4+2t_1t_2t_4+2t_2^2t_4+2t_1t_3t_4+2t_2t_3t_4+2t_3^2t_4+2t_1^2t_5+2t_1t_2t_5+2t_2^2t_5+2t_1t_3t_5+2t_2t_3t_5+2t_3^2t_5+t_1t_4t_5+t_2t_4t_5+t_3t_4t_5$;
$c_2\cdot J_{(\alpha_3)}=\{82,\ 82,\ 72,\ 24,\ 24\}$;

- phase α_{10}: $\mathcal{K}^0_{(\alpha_{10})}=6t_1^3+3t_1^2t_2+t_1t_2^2+2t_1^2t_3+t_1t_2t_3+2t_1^2t_4+t_1t_2t_4+t_1t_3t_4+2t_1^2t_5+t_1t_2t_5+t_1t_3t_5+t_1t_4t_5$;
$c_2\cdot J_{(\alpha_{10})}=\{72,\ 36,\ 24,\ 24,\ 24\}$;

- phase α_{18}: $\mathcal{K}^0_{(\alpha_{18})} = 8t_1^3 + 9t_1^2 t_2 + 9t_1 t_2^2 + 8t_2^3 + 8t_1^2 t_3 + 9t_1 t_2 t_3 + 8t_2^2 t_3 + 8t_1 t_3^2 + 8t_2 t_3^2 + 7t_3^3 + 8t_1^2 t_4 + 9t_1 t_2 t_4 + 8t_2^2 t_4 + 8t_1 t_3 t_4 + 8t_2 t_3 t_4 + 7t_3^2 t_4 + 8t_1 t_4^2 + 8t_2 t_4^2 + 7t_3 t_4^2 + 6t_4^3 + 3t_1^2 t_5 + 3t_1 t_2 t_5 + 3t_2^2 t_5 + 3t_1 t_3 t_5 + 3t_2 t_3 t_5 + 3t_3^2 t_5 + 3t_1 t_4 t_5 + 3t_2 t_4 t_5 + 3t_3 t_4 t_5 + 3t_4^2 t_5 + t_1 t_5^2 + t_2 t_5^2 + t_3 t_5^2 + t_4 t_5^2$;
  $c_2 \cdot J_{(\alpha_{18})} = \{92, 92', 82, 72, 36\}$;

Appendix 6 Gromov-Witten inv. of CY3-fold with (4,214) and a tensor

The Mori vectors in **CFPR** model side corresponding to phase 6 in [27] are given by

$$
\begin{array}{l}
(\ a_1,\ a_2,\ a_4,\ a_5,\ a_6,\ a_7,\ a_8,\ a_9\ ) \\
(\ \ 2,\ \ 1,\ \ 3,\ \ 0,\ \ 0,\ \ 1,-1,\ \ 0) = \ell_1 \\
(\ \ 0,\ \ 1,\ \ 0,\ \ 1,\ \ 0,\ \ 0,\ \ 0,-2) = \ell_2 \\
(\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 1,\ \ 1,\ \ 0,-2) = \ell_3 \\
(\ \ 0,-1,\ \ 0,\ \ 0,\ \ 0,-1,\ \ 1,\ \ 1) = \ell_4.
\end{array}
$$

The $\mathcal{K}^0_{ijkII}$ and $C_2 \cdot J_i$ of phase 6 is given in [27].

$$
\begin{aligned}
\mathcal{K}^0 &= 7t_4^3 + 2t_4^2 t_2 + 2t_4^2 t_3 + t_4 t_2 t_3 + 8t_4^2 t_1 \\
&\quad + 2t_4 t_2 t_1 + 2t_4 t_3 t_1 + t_2 t_3 t_1 + 8t_4 t_1^2 + 2t_2 t_1^2 + 2t_3 t_1^2 + 8t_1^3, \\
&\quad C_2 \cdot J = \{82, 24, 24, 92\}. \qquad (1)
\end{aligned}
$$

In this case, t_2 and t_3 are symmetric [44].

$$
K_{II}^{(g=0)\mathrm{NP}} = \frac{1}{(2\pi)^3} \sum_{n_1,n_2,n_3,n_4} N(n_1,n_2,n_3,n_4)\mathrm{Li}_3(\Pi_{i=1}^4 q_i^{n_i}).
$$

The reductions of the Mori vectors of CY3-fold to those of B_8, $S-T-U$ model and F_2 with a blow up point are as follows:

- $(\ell_1,\ell_2,\ell_3,\ell_4) \to (\ell_1+\ell_4, \ell_2-\ell_3), (\ell_1+\ell_4, \ell_2+\ell_4), (\ell_1+\ell_4, \ell_3+\ell_4)$ for B_8,
- $(\ell_1,\ell_2,\ell_3,\ell_4) \to (\ell_1+\ell_4, \ell_2, \ell_3)$ for S-T-U model with (3,243),
- $(\ell_1,\ell_2,\ell_3,\ell_4) \to_{(T^2\to\infty)} (0, \ell_2', \ell_3', \ell_4')$ for F_2 with a blow up point.

[44] $\mathcal{K}^0_{ijk\ \mathrm{II}}$ of phase f with $(h^{1,1}, h^{2,1}) = (5,185)$ can be truncated to the one in 6 phase case by setting $t_1 = 0$ and replacing $t_{i+1} \to t_i$ for $i = 2\cdots 5$, which is given by [27].

To take the heterotic string side, we use the linear transformations from t_i to S,T,U,V given in [27] : $t_1=V,\ t_2=T-U,\ t_3=S-U,\ t_4=U-V$. After taking a weak couplig limit such as $t_3\to\infty$ and $q_3=n_3=0$, the only sequences with $n_1\geq n_2$ seem to remain.

The sequences with $n_1=n_2, n_3=0$ are represented by $Z^{instB_8}_{0;n}(\tau)$ where $Z^{instB_8}_{g;n}(\tau)=\Sigma_{k|n}\mu(k)k^{-3}Z_{g,\frac{n}{k}}{}^{B_8}(k\tau)$ and $\mu(k)$ is a Möbius function.
$Z^{B_8}_{0;1}=\frac{1}{\eta^{12}}E_4$, $Z^{B_8}_{0;2}=\frac{1}{24\eta^{24}}E_4(2E_6+E_2E_4)$,
$Z^{B_8}_{0;3}=\frac{1}{15552\eta^{36}}E_4(109E_4^3+197E_6^2+216E_2E_4E_6+54E_2^2E_4^2)$, etc.
Therefore, they seem to be represented by the Dedekind eta function, $\eta(\tau)$ and the Eisenstein series, E_i only.
$Z^{instB_8}_{0;n}=Z^{inst\ B_8}_{0;n,m}q^m$:
$Z^{instB_8}_{0;1}=1+252q+5130q^2+\cdots$
$Z^{instB_8}_{0;2}=-9252q^2-673760q^3+\cdots$
$Z^{instB_8}_{0;3}=948628q^3+115243155q^4+\cdots$, etc.
The examples of the Gromov-Witten invariants with $n_1>n_2$ and $n_3=0$ that are not represented by $Z^{instB_8}_{0;n}$ are N(2,1,0,2)=265968, N(3,1,0,3)=162273760 and N(3,1,0,2)=1739160 etc [45] [46]. [47]
$\lim_{n_3=0}K_{II}^{(g=0)\mathrm{NP}}=420(\Sigma_n Li_3((q_1q_4)^n)+\Sigma_n Li_3((q_1q_2q_4)^n))$
$+\Sigma_{n,m}Z^{B_8\,inst}_{\ \ 0;n,m}(Li_3(q_1^n(q_1q_2q_4)^m)+\Sigma_{n,m}Z^{B_8\,inst}_{\ \ 0;n,m}Li_3((q_1q_2q_4^2)^n(q_1q_2q_4)^m)$
$+\Sigma_{n,m}Z^{B_8\,inst}_{\ \ 0;n,m}Li_3(q_1^n(q_1q_4)^m)+\Sigma_{n,m}Z^{B_8\,inst}_{\ \ 0;n,m}Li_3((q_1q_2q_4)^n(q_1q_4)^m))$
$+N(2,1,0,2)\mathrm{Li}_3(q_1^2q_2q_4^2)+N(3,1,0,2)\mathrm{Li}_3(q_1^3q_2q_4^2)+N(3,1,0,3)\mathrm{Li}_3(q_1^3q_2q_4^3)+\cdots$.

[45] However, 265968 and 162273760 exist in the Gromov-Witten invariants of CY3-fold with (4,214) and $(k_1,k_2)=(11,13)$. It is χ_0 with a extra vector multiplet and K3=$\mathbf{P}^3(1,1,3,5)[12]$ fiber that is given by [28]. In χ_0 case, both $\Sigma_{n_4}N(n_1,0,n_3,n_4)$ and $\Sigma_{n_4}N(n_1,n_2,0,n_4)$ lead to the coefficient of $\frac{2E_4E_6}{\eta^{24}}$ expansion though S and T are not symmetric, because they reduce to $(h^{1,1},h^{1,2})=(3,243)$ case where S and T are symmetric. N(2,0,1,4)=265968 and N(3,0,1,6)=162273760 correspond to the limit of $T\to\infty$. They are in the contribution from the non-perturbative vector multiplet for taking S as the dilaton.

[46] 1739160 exists in the Gromov-Witten invariants of $(h^{1,1},h^{1,2})=(5,185)$ with two tensors case such as the phase f. In this case, almost Gromov-Witten invariants are represented by those of B_8. The others relate to those of the phase 16 of the list that is given by [27]. The phase 16 has a 6-dim tensor and a 6-dim vector[45].

[47] The phase 14 of the list that is given by [27] coinsides with the perturbative coupling such as $\frac{E_{4,1}(r_1,\tau)E_{6,1}(r_2,\tau)}{\eta^{24}}+\frac{E_{4,1}(r_2,\tau)E_{6,1}(r_1,\tau)}{\eta^{24}}$. which is the N=2 model with two 6-dim. vector multiplets[45].

The partition function of the model with a nonperturbative vector for $T \to \infty$ will be also represented by the quasi modular forms and the character of the Kac-Moody algebra including E_2 and a Wilson line. By examining their relations and taking an appropriate limit, the partition function with a tensor will be also represented in the quasi modular forms and the characters.

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	models	(I)	$G_1 = I$	(II)	$G_1 = A_1$	(II)	$G_1 = A_2$
n^0	G_2	$(h^{1,1}, h^{2,1})$	CY3 weight	$(h^{1,1}, h^{2,1})$	CY3 weight	$(h^{1,1}, h^{2,1})$	CY3 weight
0	I	$(3, 243)$		$(4, 214)$		$(5, 197)$	
2	I	$(3, 243)$	$(1, 1, 2, 8, 12)$	$(4, 190)$	$(1, 1, 2, 6, 10)$	$(5, 161)$	$(1, 1, 2, 6, 8)$
3	A_2	$(5, 251)$	$(1, 1, 3, 10, 15)$	$(6, 186)$	$(1, 1, 3, 7, 12)$	$(7, 151)$	$(1, 1, 3, 7, 9)$
4	D_4	$(7, 271)$	$(1, 1, 4, 12, 18)$	$(8, 194)$	$(1, 1, 4, 8, 14)$	$(9, 153)$	$(1, 1, 4, 8, 10)$
6	E_6	$(9, 321)$	$(1, 1, 6, 16, 24)$	$(10, 220)$	$(1, 1, 6, 10, 18)$	$(11, 167)$	$(1, 1, 6, 10, 12)$
8	E_7	$(10, 376)$	$(1, 1, 10, 24, 36)$	$(11, 267)$	$(1, 1, 10, 14, 26)$	$(12, 186)$	$(1, 1, 10, 14, 16)$
12	E_8	$(11, 491)$	$(1, 1, 12, 28, 42)$	$(12, 318)$	$(1, 1, 12, 16, 30)$	$(13, 229)$	$(1, 1, 12, 16, 18)$

Table 1: type IIA-heterotic string duality : Hodge and instanton numbers of CY3s in (I) and (II) in A-chain

n^0	G_2	$h^{1,1}$	$h^{2,1}$	k_1	k_2	n_T^0	Δn_T	n_T	$K3$ fiber
0	I	3	243	12	12	1	0	1	$\mathbf{P}^3(1, 1, 4, 6)[12]$
2	I	3	243	$12 + 2$	$12 - 2$	1	0	1	$\mathbf{P}^3(1, 1, 4, 6)[12]$
3	A_2	5	251	$12 + 3$	$12 - 3$	1	0	1	$\mathbf{P}^3(1, 2, 6, 9)[18]$
4	D_4	7	271	$12 + 4$	$12 - 4$	1	0	1	$\mathbf{P}^3(1, 2, 6, 9)[18]$
6	E_6	9	321	$12 + 6$	$12 - 6$	1	0	1	$\mathbf{P}^3(1, 3, 8, 12)[24]$
8	E_7	11	376	$12 + 8$	$12 - 8$	1	0	1	$\mathbf{P}^3(1, 4, 10, 15)[30]$
12	E_8	12	491	$12 + 12$	$12 - 12$	1	0	1	$\mathbf{P}^3(1, 5, 12, 18)[36]$

Table 2: Hodge and instanton numbers of CY3s in (I)

n^0	$G_2 \times G_1$	$h^{1,1}$	$h^{1,2}$	k_1	k_2
0	$I \times A_1$	4	214	12	12
1	$I \times A_1$	4	202	$12+1$	$12-1$
2	$A_2 \times A_1$	4	190	$12+2$	$12-2$
6	$E_6 \times A_1$	6	220	$12+6$	$12-6$
8	$E_7 \times A_1$	11	251	$12+8$	$12-8$
10	$E_8 \times A_1$	14	284	$12+10$	$12-10$

Table 3: Hodge and instanton numbers of CY3s in (II) with $G_1 = A_1$ in A series

Δn_T	G_2	$h^{1,1}$	$h^{2,1}$	k_1	k_2	n_T^0	n_T	$\Delta h^{1,1}$	$\Delta h^{2,1}$
0	I	3	243	12	12	1	1	0	0
2	I	5	185	12	$12-2$	1	3	2	-58
3	A_2	8	164	12	$12-3$	1	4	3	-87
4	D_4	11	155	12	$12-4$	1	5	4	-116
6	E_6	15	147	12	$12-6$	1	7	6	-174
8	E_7	18	144	12	$12-8$	1	9	8	-132
12	E_8	23	143	12	$12-12$	1	13	12	-348

Table 4: The Hodge and instanton numbers in (III)/(IV)

CY3s in (III)			CY3s in (IV)		
Δn_T	K3	K3	s	weight	K3
0	$\mathbf{P}^3(1,1,4,6)[12]$	$\mathbf{P}^3(1,1,4,6)[12]$	1		$\mathbf{P}^3(1,1,4,6)[12]$
2	$\mathbf{P}^3(1,1,4,6)[12]$	$\mathbf{P}^3(1,1,4,6)[12]$	2	$(1,1,2,8,12)$	$\mathbf{P}^3(1,1,4,6)[12]$
3	$\mathbf{P}^3(1,1,4,6)[12]$	$\mathbf{P}^3(1,2,6,9)[18]$	3	$(1,2,3,12,18)$	$\mathbf{P}^3(1,1,4,6)[12]$
4	$\mathbf{P}^3(1,1,4,6)[12]$	$\mathbf{P}^3(1,2,6,9)[18]$	4	$(1,4,5,20,30)$	$\mathbf{P}^3(1,1,4,6)[12]$
6	$\mathbf{P}^3(1,1,4,6)[12]$	$\mathbf{P}^3(1,3,8,12)[24]$	6	$(1,6,7,28,42)$	$\mathbf{P}^3(1,1,4,6)[12]$
8	$\mathbf{P}^3(1,1,4,6)[12]$	$\mathbf{P}^3(1,4,10,15)[30]$	8	$(1,8,9,36,54)$	$\mathbf{P}^3(1,1,4,6)[12]$
12	$\mathbf{P}^3(1,1,4,6)[12]$	$\mathbf{P}^3(1,5,12,18)[36]$	12	$(1,12,13,52,78)$	$\mathbf{P}^3(1,1,4,6)[12]$

Table 5: The type of K3 sub dual polyhedra contained in (III) and case (IV)

s	$G_2 \times G_1$	$h^{1,1}$	$h^{2,1}$	$\Delta h^{1,1}$	$\Delta h^{2,1}$	k_1	k_2	Δn_T
3	$I \times A_1$	9	129	5	-85			
2	$I \times A_1$	6	144	2	$-58 = -29 \times 2$	$12 + 1$	$12 - 1 - 2$	2
1	$A_1 \times A_1$	4	190	0	0	$12 + 2$	$12 - 2$	0
5	$E_6 \times A_1$	16	118	6	$-102 = -17 \times 6$	$12 + 6 - 6$	$12 - 6$	6
7	$E_7 \times A_1$	19	115	8	$-136 = -17 \times 8$	$12 + 8 - 8$	$12 - 8$	8
9	$E_8 \times A_1$	24	114	10	$-170 = -17 \times 10$	$12 + 10 - 10$	$12 - 10$	10

Table 6: The Hodge numbers of P^4(1,s,(1+s)(1,3,5))[10s] and the relation of heterotic duality for (VI)with $G_1 = A_1$ in A series.

model	#{K3 fibrations}	#{phases by triangulation}
(I†)	$\{0, 1, 2\}$	8 phases
(III)	$\{0, 1, 2\}$	8 phases labeled by a, …, h
(IV)	$\{0, 1\}$	5 phases labeled by A, …, E
(V)	$\{0, 1, 2, 3\}$	18 phases labeled by $\alpha_1, \ldots, \alpha_{18}$

Table 7: The number of K3 fibrations and the phases specified by the triangulations in four models

No of B_8 in (A)	$N = 252$	$N = -9252$	$N = 848628$
1 $\{n_i\}$	10001	20002	30003
2 $\{n_i\}$	10011	20022	30033
3 $\{n_i\}$	11101	22202	33303
4 $\{n_i\}$	11111	22222	33333
5 $\{n_i\}$	11001	22002	33003
6 $\{n_i\}$	11011	22022	33033
No of B_8 in (6)	$N = 252$	$N = -9252$	$N = 848628$
1 $\{n_i\}$	1000	2000	3000
2 $\{n_i\}$	1012	2024	3036
3 $\{n_i\}$	1102	2204	3306

Table 8: $N(\{n_i\})$ of phase A with $(h^{1,1}, h^{2,1}) = (5, 185)$ which denote B_8 and $N(\{n_i\})$ of phase 6 with $(h^{1,1}, h^{2,1}) = (4, 214)$ which denote B_8

No of $Z^{\mathrm{inst}B8}_{0;0,m}$ in (6)	$N = 252$	$N = 5130$
1 $\{n_i\}$	1000	2001
2 $\{n_i\}$	1000	2011
3 $\{n_i\}$	1000	2101
4 $\{n_i\}$	1012	2013
5 $\{n_i\}$	1102	2103
6 $\{n_i\}$	1102	2203
7 $\{n_i\}$	1012	2023

Table 9: $N(\{n_i\})$ of phase 6 with $(h^{1,1}, h^{2,1}) = (4, 214)$ which denote B_8

$\sharp$\{K3 fibrations\}	$s = 2$ in (IV)	$\Delta n_{\mathrm{T}} = 2$ in (III)	$\Delta n_{\mathrm{T}} = 2$ in (V)
1	A		
1	B	c	
0	C	d	α_{14}
1	D	b	
1	E	a	$\alpha_2, \alpha_7, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{17}$
2		e	$\alpha_1, \alpha_4, \alpha_8, \alpha_9, \alpha_{15}, \alpha_{16}$
2		f	α_5, α_6
2		g	
1		h	
3			α_{10}
1			α_{18}
0			α_3

Table 10: Identification of phases. The phases in the same line are the same by criterion 1

phase	$c_2 \cdot J_i$	equivalent phase
A	$\{72, 48, 24, 36, 72'\}$	g, α_{10}
$a = E$	$\{72, 82, 24, 92, 36\}$	h
$b = D$	$\{72, 92, 24, 82, 48\}$	f, α_3
$c = B$	$\{72, 82, 24, 36, 48\}$	e
$d = C$	$\{72, 82, 92, 102, 36\}$	α_{18}
e	$\{72, 36, 24, 24, 82\}$	$c = B$
f	$\{72, 82, 24, 24, 92\}$	$b = D, \alpha_3$
g	$\{72, 24, 36, 24, 48\}$	A, α_{10}
h	$\{36, 24, 82, 92, 82'\}$	$a = E$
α_3	$\{82, 82, 72, 24, 24\}$	$f, b = D$
α_{10}	$\{72, 36, 24, 24, 24\}$	A, g
α_{18}	$\{92, 92', 82, 72, 36\}$	$d = C$

Table 11: The relation of phases by criterion 2. Phases in the same line denote the equivalent phases. There are only five phases, A,B,C,D, and E of (IV) in **HLY** model which are topologically non equivalent.